ISRAEL JOURNAL OF MATHEMATICS, Vol. 68, No. 1, 1989

ON l^p-COMPLEMENTED COPIES IN ORLICZ SPACES II†

BY

FRANCISCO L. HERNÁNDEZ AND BALTASAR RODRIGUEZ-SALINAS
Dpto. Análisis Matemático, Facultad de Matemáticas,
Universidad Complutense, 28040 — Madrid, Spain

ABSTRACT

For any $p > 1$, the existence is shown of Orlicz spaces L^F and l^F with indices p containing *singular* l^p-complemented copies, extending a result of N. Kalton ([6]). Also the following is proved: *Let* $1 < \alpha \leqq \beta < \infty$ *and H be an arbitrary closed subset of the interval* $[\alpha, \beta]$. *There exist Orlicz sequence spaces* l^F *(resp. Orlicz function spaces* L^F*) with indices* α *and* β *containing only singular* l^p*-complemented copies and such that the set of values* $p > 1$ *for which* l^p *is complementably embedded into* l^F *(resp.* L^F*) is exactly the set H (resp.* $H \cup \{2\}$*).* An explicitly defined class of minimal Orlicz spaces is given.

Introduction

The class of *minimal* Orlicz sequence spaces was introduced by J. Lindenstrauss and L. Tzafriri in ([8], [9]) proving the existence of reflexive Orlicz sequence spaces l^F containing no complemented subspaces isomorphic to l^p for any $p \geqq 1$. An extension of the notion of minimality to the context of Orlicz function spaces $L^F(\mu)$ was given in [1]. The examples of minimal Orlicz functions F obtained until today have *not* been *explicitly defined*, excluding the trivial multiplicative ones. Indeed, the existence of minimal functions is proved with the help of Zorn Lemma and all known examples are obtained, up to equivalence, via a sophisticated method by constructing Orlicz functions F_ρ associated with 0–1 valued sequences $\rho = (\rho(n))$ developed by J. Lindenstrauss and L. Tzafriri ([8], [9], [10]).

One of the purposes of this paper is to show a concrete class of minimal

† Supported in part by CAICYT grant 0338–84.
Received December 20, 1988 and in revised form March 2, 1989

Orlicz spaces $L^F(\mu)$, where the minimal functions F are *explicitly defined* in terms of elementary functions. (As far as we know these functions are the first examples explicitly presented.)

A second purpose of this paper is to study the following "inverse problem" for *singular l^p-complemented copies*:

In [6] N. J. Kalton proved the existence of Orlicz sequence spaces l^F with different indices containing an l^p-complemented copy (for $p > \frac{3}{2}$) and such that the function t^p does *not* belong to the set $E_{F,1}$. In other words, no complemented subspace isomorphic to l^p in l^F is generated by a block basis with constant coefficients of the canonical basis of l^F. (In short, we shall say that l^F *has a singular l^p-complemented copy*.) Thus, the "*inverse problem*" for singular l^p-complemented copies can be stated as follows: Given an arbitrary set H of real numbers $p > 1$, find Orlicz sequence spaces l^F containing only singular l^p-complemented copies and such that the set of values $p \geqq 1$ for which l^p is complementably embedded into l^F is exactly the prefixed set H.

Let us mention that the corresponding inverse problem for "*natural*" l^p-complemented copies (i.e. when the functions $t^p \,\tilde{\in}\, E_{F,1}$) has previously been solved in [3] for arbitrary closed sets H.

In Section I we show that the class of functions $F_{p,q}(t) = t^p \exp\{qf(\log t)\}$, where f is the function $f(x) = \sum_{k=1}^{\infty} (1 - \cos(\pi x/2^k))$, $p > 1$ and q arbitrary, is minimal. In particular, we get that the functions $F_{p,1}$, $p > 1$, studied by W. Johnson, B. Maurey, G. Shechtman and L. Tzafriri in ([4] Ch. 8) are minimal. Theorem 1.6 gives a necessary analytic criterion, in terms of an oscillation constant γ_f associated with the function f, for the embedding of l^p into the Orlicz space $L^{F_{p,q}}$ as a complemented subspace. Among other consequences, we easily deduce (Corollary 1.7) a result due to Lindenstrauss and Tzafriri obtained by making use of the method of 0–1 valued sequences: The existence of reflexive Orlicz sequence spaces l^F with indices p containing no l^p-complemented copy.

Section II is devoted to *singular* l^p-complemented copies in Orlicz sequence spaces l^F, answering the "*inverse problem*" for closed sets (Theorems 2.2 and 2.3): Given $1 < \alpha \leqq \beta < \infty$ and an arbitrary closed subset H of the interval $[\alpha, \beta]$, there exists an Orlicz sequence space l^F with indices $\alpha_F = \alpha$ and $\beta_F = \beta$ containing *only* singular l^p-complemented copies and such that the set of values p's for which l^p is complementably embedded into l^F is exactly the set H.

In Section III we show the existence of Orlicz function spaces $L^F(0, 1)$ containing *singular* l^p-complemented copies (i.e. $t^p \notin E_{F,1}^{\infty}$), extending the above-mentioned result of Kalton ([6] p. 276) for Orlicz sequence spaces: For any $p > 1$ there exist Orlicz function spaces $L^F(0, 1)$ having complemented subspaces isomorphic to l^p and none of these subspaces is generated by a sequence of pairwise disjoint characteristic functions. Previously, to get this, we study when the inclusion map between Orlicz function spaces is a *disjointly singular* operator (a notion defined below, which is weaker than the strict singularity). Finally, as an application, we solve also in this context of Orlicz function spaces the "inverse problem" for *singular* l^p-complemented copies and arbitrary closed sets (Theorem 3.6).

Preliminaries

Given a positive measure space (Ω, Σ, μ) and an Orlicz function F (i.e. a continuous convex non-decreasing function defined for $x \geqq 0$ so that $F(0) = 0$ and $F(1) = 1$), the *Orlicz function space* $L^F(\mu)$ is defined as the set of equivalence classes of μ-measurable scalar functions u of (Ω, Σ, μ) such that

$$|ru|_F = \int_\Omega F(r|u|)d\mu < \infty, \qquad \text{for some } r > 0.$$

The space $L^F(\mu)$ endowed with the Luxemburg norm

$$\| u \| = \inf\{r > 0 : |u/r|_F \leqq 1\}$$

is a Banach space. We shall consider as measure spaces the $(0, 1)$ and $(0, \infty)$ intervals with the Lebesgue measure, writing then $L^F(0, 1)$ and $L^F(0, \infty)$. Similarly, the *Orlicz sequence space* l^F consists of all those sequences $u = (u_n)$ of scalars for which there is an $r > 0$ with $|ru|_F = \Sigma_{n=1}^\infty F(r|u_n|) < \infty$.

We assume that the Orlicz function F satisfies the Δ_2-condition at ∞ and at 0, so the associated indices verify $1 \leqq \alpha_F^\infty \leqq \beta_F^\infty < \infty$ and $1 \leqq \alpha_F \leqq \beta_F < \infty$ (cf. [10], [11]). We shall consider the following compact subsets related to F in the space of the continuous functions $C(0, 1)$ and in the space $C(0, \infty)$ endowed with the compact-open topology:

$$E_{F,s} = \overline{\left\{\frac{F(rt)}{F(r)} : r \leqq s\right\}}; \qquad E_F = \bigcap_{s>0} E_{F,s},$$

$$E_{F,s}^\infty = \overline{\left\{\frac{F(rt)}{F(r)} : r \geqq s\right\}}; \qquad E_F^\infty = \bigcap_{s>0} E_{F,s}^\infty,$$

$$C_{F,s} = \overline{\text{conv}}\, E_{F,s}; \qquad C_{F,s}^\infty = \overline{\text{conv}}\, E_{F,s}^\infty,$$

for every $s > 0$.

An Orlicz function F is called *minimal* at 0 ([8], [9]) if for every function $G \in E_{F,1}$, as a subset of $C(0, 1)$, we have $E_{G,1} = E_{F,1}$. The existence of minimal functions, different from the t^p, is proved by Zorn Lemma. In ([9], [10] p. 164) it was proved that there are minimal Orlicz sequence spaces l^F with arbitrary indices containing no l^p-complemented subspaces for any $p \geqq 1$. To show this J. Lindenstrauss and L. Tzafriri developed a method of constructing Orlicz functions F_ρ associated to a sequence of digits $(\rho(n))$ with $\rho(n)$ equal to 0 or 1, and characterizing the minimal functions of the form F_ρ.

Minimal Orlicz function spaces L^F were introduced in ([1]): An Orlicz function F is called *minimal* (at ∞) if for every function $G \in E^\infty_{F,1}$, as a subset of $C(0, \infty)$, we have $E^\infty_{G,1} = E^\infty_{F,1}$. Any minimal Orlicz function space $L^F(0, 1)$ contains a complemented copy of l^F. In ([1], [3]) the existence of minimal function spaces $L^F(0, 1)$ was proved with arbitrary indices containing no l^p-complemented copies for any $p \neq 2$. A criterion to insure that reflexive Orlicz function spaces $L^F(0, 1)$ contain no complemented copies of l^p $(p \neq 2)$ is that the function t^p be strongly non-equivalent to $E^\infty_{F,1}$ ([3] Thm. 4).

We refer to ([10], [11], [12]) for other definitions and terminology used on Orlicz and Banach spaces.

I. An explicitly defined class of minimal Orlicz spaces

In this section we present an explicit class of Orlicz spaces with the property of being minimal. The motivation for this class is to be found in the functions defined by W. Johnson, B. Maurey, G. Schechtman and L. Tzafriri in ([4], p. 235), $F(t) = t^p \exp\{f(\log t)\}$, where f means the function

$$f(x) = \sum_{k=1}^{\infty} (1 - \cos(\pi x/2^k))$$

and $p > 1$. They proved, answering a problem of Mityagin, that the Orlicz function spaces $L^F(0, 1)$ and $L^F(0, \infty)$ are isomorphic.

Lemma 1.1. *Given scalars $p > 1$ and q arbitrary, the function $F_{p,q}$ defined by $F_{p,q}(0) = 0$ and*

$$F_{p,q}(t) = t^p e^{qf(\log t)}, \qquad \text{if } t > 0,$$

is equivalent to a convex function and its associated indices are equal to p. Furthermore if $|q| < (p-1)/3\pi$ then $F_{p,q}$ is itself convex.

PROOF. Consider $F_{p,q} \equiv F$ for $q \neq 0$ (otherwise the result is obvious). We know ([4], p. 237) that for every $\varepsilon > 0$ there is a constant K_ε such that

$$(+) \qquad |f(s+t) - f(s)| \leqq \varepsilon |t| + K_\varepsilon$$

for all $s, t \in \mathbf{R}$. Since

$$\frac{F(st)}{F(s)} = t^p \exp\{q[f(\log s + \log t) - f(\log s)]\},$$

we deduce that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that for every $s > 0$,

$$C_\varepsilon^{-1} t^{p - |q|\varepsilon} \leqq \frac{F(st)}{F(s)} \leqq C_\varepsilon t^{p + |q|\varepsilon} \qquad \text{if } t > 1,$$

and if $0 < t < 1$,

$$C_\varepsilon^{-1} t^{p + \varepsilon|q|} \leqq \frac{F(st)}{F(s)} \leqq C_\varepsilon t^{p - |q|\varepsilon}.$$

This implies that F is equivalent to a convex function and with associated indices

$$\alpha_F = \alpha_F^\infty = \beta_F = \beta_F^\infty = p.$$

Now, if $|q| < (p-1)/3\pi$ we have that

$$|f'(x)| \leqq \sum_{k=1}^{\infty} \frac{\pi}{2^k} \left|\sin \frac{\pi x}{2^k}\right| \leqq \pi$$

and

$$|f''(x)| \leqq \sum_{k=1}^{\infty} \frac{\pi^2}{2^{2k}} \left|\cos \frac{\pi x}{2^k}\right| \leqq \frac{\pi^2}{3} \leqq 2\pi.$$

Thus we get, with $H(t)$ denoting the function $qf(\log t)$, that

$$\begin{aligned} F''(t) t^{2-p} e^{-H(t)} &= p(p-1) + (2p-1) q f'(\log t) + q^2 (f'(\log t))^2 + q f''(\log t) \\ &\geqq p(p-1) - (2p+1)|q|\pi \geqq 0. \end{aligned}$$

Therefore F is a convex function q.e.d.

The following result shows that all functions $F_{p,q}$ are minimal. In particular we get that the Johnson et al. function is minimal.

THEOREM 1.2. *Given $p > 1$ and q arbitrary, the Orlicz spaces $L^{F_{p,q}}(\mu)$ and $l^{F_{p,q}}$ are minimal.*

PROOF. Let us show that $F_{p,q} \equiv F$ for $q \neq 0$ is minimal at ∞. (For $q = 0$ the result is obvious.) If $G \in E_{F,1}^{\infty}$ and G is not equivalent to F, there exists a sequence $(s_n) \nearrow \infty$, such that

$$G(t) = \lim_{n\to\infty} \frac{F(e^{s_n}t)}{F(e^{s_n})} = t^p e^{qg(\log t)}$$

uniformly on the compact subsets of $[0, \infty)$ and where g is the function defined by

$$g(x) = \lim_{n\to\infty} (f(s_n + x) - f(s_n))$$

$$= \lim_{n\to\infty} \sum_{k=1}^{\infty} \left(\cos\frac{\pi s_n}{2^k} - \cos\frac{\pi(x + s_n)}{2^k}\right).$$

For each $m \in \mathrm{N}$ we can take a scalar $0 \leqq s_n^{(m)} \leqq 2^{m+1}$ such that $s_n^{(m)} \equiv s_n$ (mod 2^{m+1}). So, there exists a subsequence of $(s_n^{(m)})_{n=1}^{\infty}$ converging to a $\sigma_m \in [0, 2^{m+1}]$. Using the Cantor Diagonal method, we obtain a subsequence, denoted also by (s_n), such that $s_n^{(m)} \to \sigma_m$ and $0 \leqq \sigma_m \leqq 2^{m+1}$ for each $m \in \mathrm{N}$.

Now, by using the Weierstrass criterion on uniform convergence, and an iterated limit argument (see e.g. [13] p. 198), it results that

$$g(x) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \left(\cos\frac{\pi s_n^{(k)}}{2^k} - \cos\frac{\pi(x + s_n^{(k)})}{2^k}\right)$$

$$= \sum_{k=1}^{\infty} \left(\cos\frac{\pi\sigma_k}{2^k} - \cos\frac{\pi(x + \sigma_k)}{2^k}\right).$$

If we consider the sequence $(r_n) = (2^{n+1} - \sigma_n)$ we get that

$$\lim_{n\to\infty} (g(r_n + x) - g(r_n)) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \left(\cos\frac{(r_n + \sigma_k)\pi}{2^k} - \cos\frac{(x + r_n + \sigma_k)\pi}{2^k}\right)$$

$$= \sum_{k=1}^{\infty} \left(1 - \frac{\cos \pi x}{2^k}\right) = f(x)$$

by using again the Weierstrass criterion and that $\sigma_n \equiv \sigma_k$ (mod 2^{k+1}) for any $n \geqq k$.

Thus

$$\lim_{n\to\infty}\frac{G(e^{r_n}t)}{G(r^{r_n})}=t^p\left(\lim_{n\to\infty}e^{q[g(r_n+\log t)-g(r_n)]}\right)$$

$$=t^pe^{qf(\log t)}=F(t)$$

uniformly on compact subsets of $[0,\infty)$. So $F\in E^{\infty}_{G,1}$, which implies that $E^{\infty}_{F,1}\subset E^{\infty}_{G,1}\subset E^{\infty}_{F,1}$. Hence $E^{\infty}_{F,1}=E^{\infty}_{G,1}$ and F is a minimal function at ∞.

Finally, the minimality of the function $F_{p,q}$ at 0 follows from Proposition 1 in [1]. q.e.d.

REMARK 1. From the above proof it follows that a function $G\in E^{\infty}_{F,1}$ if and only if $G(t)=t^p\exp\{qg_\sigma(\log t)\}$, where g_σ is the function

$$g_\sigma(x)=\sum_{k=1}^{\infty}\left(\cos\frac{\pi\sigma_k}{2^k}-\cos\frac{\pi(x+\sigma_k)}{2^k}\right)$$

and $\sigma=(\sigma_k)_{k=1}^{\infty}$ is a scalar sequence satisfying $0\leqq\sigma_k<2^{k+1}$ and $\sigma_n\equiv\sigma_k$ (mod 2^{k+1}) for $n\geqq k$.

In particular, these functions G are also examples of explicitly defined minimal functions.

REMARK 2. Fix $p>1$; the Orlicz spaces $L^{F_{p,q}}(\mu)$ and $L^{F_{p,r}}(\mu)$ are not isomorphic for any parameters $q\neq r$. Indeed, let us suppose that both spaces are isomorphic. Then, from Theorem 7.1 in ([4]) we deduce that the functions $F_{p,q}$ and $F_{p,r}$ are equivalent. But this is not possible since the function $f(x)$ is not bounded at $\pm\infty$.

Also it can be shown, by reasoning as in ([4] p. 236), that the Orlicz spaces $L^{F_{p,q}}(0,1)$ and $L^{F_{p,q}}(0,\infty)$ are Riesz-isomorphic for any $p>1$ and q arbitrary.

We study now the embedding of l^p into the spaces $L^{F_{p,q}}$ as complemented subspaces. We shall show that inside this class of spaces $L^{F_{p,q}}$ there are spaces without complemented copies of l^p-spaces. In order to show it, we need to introduce a slight variant of the notion of strong non-equivalence given by Lindenstrauss and Tzafriri ([9], [10] p. 150).

DEFINITION. Let F be an Orlicz function and a scalar $\sigma>0$. An Orlicz function is called *σ-strongly non-equivalent* to $E_{F,1}$ (resp. $E^{\infty}_{F,1}$) if there exist two sequences of numbers (K_n) and integers (m_n), with $\lim K_n=\infty$, $m_n=o(K_n^\sigma)$, and m_n-points $t_i\in(0,1)$ such that for every $\lambda\in(0,1)$ (resp. $\lambda\in[\max_i t_i^{-1},\infty)$) there is at least one index i, $1\leqq i\leqq m_n$ for which

$$\frac{F(\lambda t_i)}{F(\lambda)G(t_i)} \notin \left[\frac{1}{K_n}, K_n\right].$$

Given an Orlicz sequence space l^F and $0 < \sigma < 1/\beta_F$, we can assume w.l.o.g. that

$$\frac{tF'(t)}{F(t)} \leqq \frac{1}{\sigma}$$

for every $t > 0$. Hence $F(st) \leqq s^{1/\sigma}F(t)$ for every $t > 0$ and $s \geqq 1$. Now, reasoning as in the proof of Theorem 4.b.5. in ([10]) we obtain the following

PROPOSITION 1.3. *Let l^F be a separable Orlicz sequence space. If G is an Orlicz function σ-strongly non-equivalent to $E_{F,1}$ for some $\sigma < 1/\beta_F$, then l^G is not isomorphic to any complemented subspace of l^F.*

For Orlicz function spaces we can use a similar trick to obtain an extension of Theorem 4 in ([3]). In particular we have

PROPOSITION 1.4. *Let $L^F(0, 1)$ be a reflexive Orlicz function space. If t^p, for $p \neq 2$, is σ-strongly non-equivalent to $E^\infty_{F,1}$ for some $\sigma < 1/\beta^\infty_F$, then $L^F(0, 1)$ does not contain any complemented copy of l^p.*

To apply these results to the class of minimal spaces $L^{F_{p,q}}$ we need to consider an average oscillation constant γ_f associated with the function $f(x) = \Sigma_{k=1}^\infty (1 - \cos(\pi x/2^k))$.

Fix $s > 0$; let us consider

$$M_n(s) = \max_{0 \leqq x \leqq 2^n} [f(x+s) - f(s)], \qquad m_n(s) = \min_{0 \leqq x \leqq 2^n} [f(x+s) - f(s)]$$

and the oscillation

$$\omega_n^f(s) = M_n(s) - m_n(s) = \max_{0 \leqq x, y \leqq 2^n} [f(x+s) - f(y+s)],$$

If $\gamma_n^f = \inf_{s>0} \omega_n^f(s)$, the *average oscillation constant* is defined by

$$\gamma_f = \varliminf_{n \to \infty} \frac{\gamma_n^f}{n}.$$

LEMMA 1.5. *For each $n \in \mathbb{N}$ it holds that*

$$\left(1 + 2\,\mathrm{sen}\,\frac{\pi}{8}\right)\frac{(n-2)}{4} - 2\pi \leqq \gamma_n^f \leqq 2n + 4\pi.$$

Therefore the average oscillation constant γ_f satisfies

$$0 < \tfrac{1}{4}\left(1 + 2\sin\frac{\pi}{8}\right) \leqq \gamma_f \leqq 2.$$

PROOF. As the function f satisfies ([4] p. 237)

$$|f(s+t) - f(s)| \leqq 2\log_2^+|t| + 4\pi \qquad \text{for } t \geqq 0, \quad s \in \mathbf{R},$$

we get that $\omega_n^f(s) \leqq 2n + 4\pi$. Thus $\gamma_f \leqq 2$.

Now fix $n \in \mathbf{N}$ and $s > 0$; let us suppose that $\Sigma_{k=1}^{n-1} \cos(\pi s/2^k) \leqq 0$. Taking $x \in [0, 2^n]$ such that $x + s \equiv 0 \pmod{2^n}$ we have

$$\begin{aligned} f(x+s) - f(s) &\leqq -\sum_{k=1}^{n-1} \cos\frac{\pi(x+s)}{2^k} + \sum_{k=n}^{\infty}\left(\cos\frac{\pi s}{2^k} - \cos\frac{\pi(x+s)}{2^k}\right) \\ &\leqq -\sum_{k=1}^{n-1} \cos\frac{\pi(x+s)}{2^k} + 2\pi = -n + 1 + 2\pi. \end{aligned}$$

If we assume that $\Sigma_{k=1}^{n-1} \cos(\pi s/2^k) > 0$, we get

$$(*) \qquad \begin{aligned} f(x+s) - f(s) &\geqq -\sum_{k=1}^{n-1} \cos\frac{\pi(x+s)}{2^k} + \sum_{k=n}^{\infty}\left(\cos\frac{\pi s}{2^k} - \cos\frac{\pi(x+s)}{2^k}\right) \\ &\geqq -\sum_{k=1}^{n-1} \cos\frac{\pi(x+s)}{2^k} - 2\pi. \end{aligned}$$

With an $x \in [0, 2^n]$ such that $x + s \equiv (1 + 4 + \cdots + 4^n) \pmod{2^n}$ we obtain that

$$\cos\frac{\pi(x+s)}{2^{2k}} = \cos\pi\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^k}\right) \leqq \cos\pi\left(1 + \frac{1}{3}\right) = -\frac{1}{2}$$

for $1 \leqq 2k < n$, and

$$\cos\frac{\pi(x+s)}{2^{2k+1}} = \cos\frac{\pi}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^k}\right) \leqq \cos\frac{\pi}{2}\left(1 + \frac{1}{4}\right) = -\sin\frac{\pi}{8}$$

for $1 \leqq 2k + 1 < n$. Thus, from (*) we deduce

$$f(x+s) - f(x) \geqq \left(1 + 2\sin\frac{\pi}{8}\right)\left(\frac{n-2}{4}\right) - 2\pi.$$

Finally, in both cases

$$\omega_n^f(s) = M_n(s) - m_n(s) \geqq \left(1 + 2\sin\frac{\pi}{8}\right)\left(\frac{n-2}{4}\right) - 2\pi,$$

which implies the result. q.e.d.

THEOREM 1.6. *If $p > 1$ and $q \neq 0$ satisfy*

$$(+) \qquad \frac{p}{|q|} < \frac{\gamma_f}{2\log 2},$$

then the Orlicz sequence space $l^{F_{p,q}}$ does not contain any complemented copy of l^p.

PROOF. Let us prove that the function t^p is σ-strongly non-equivalent to $E_{F,1}$ for some $\sigma > 0$ and $F \equiv F_{p,q}$. For each $n \in \mathbf{N}$ put $m(n) = 2^n$ and assume the existence of an integer k verifying

$$e^{-\delta n} \leqq \frac{F(\tau^k \tau^i)}{F(\tau^k)\tau^{pi}} \leqq e^{\delta n}$$

for $\tau = e^{-1}$, $i = 1, 2, \ldots, 2^n$ and $\delta > 0$. This implies that

$$(*) \qquad -\delta n \leqq q(f(k+i) - f(k)) \leqq \delta n.$$

Now, remembering that the function f satisfies $|f(x+h) - f(x)| \leqq \pi$ for $x \geqq 0$ and $0 \leqq h \leqq 1$, we easily get the existence of integers i, j with $1 \leqq i$, $j \leqq 2^n$ such that

$$f(k+i) - f(k) \geqq M_n(k) - \pi$$

and

$$f(k+j) - f(k) \leqq m_n(k) + \pi.$$

Hence

$$f(k+i) - f(k+j) \geqq M_n(k) - m_n(k) - 2\pi \geqq \gamma_n^f - 2\pi.$$

This implies that $f(k+i) - f(k) \geqq \frac{1}{2}(\gamma_n^f - 2\pi)$ or $f(k+j) - f(k) \leqq -\frac{1}{2}(\gamma_n^f - 2\pi)$, and from $(*)$ we obtain that

$$\delta \geqq |q| \frac{(\gamma_n^f - 2\pi)}{2n} \qquad \text{for each } n \in \mathbf{N},$$

and

$$\delta \geqq |q| \frac{\gamma_f}{2}.$$

Thus, it results that for $0 < \delta < |q|\gamma_f/2$ and a big enough natural number n, for any $k \in \mathbf{N}$ there exists an integer i with $1 \leqq i \leqq 2^n = m(n)$, such that

$$\frac{F(\tau^k \tau^i)}{F(\tau^k)\tau^{pi}} \notin [e^{-\delta n}, e^{\delta n}].$$

Furthermore, by (+) we can take δ satisfying

$$\frac{2\log 2}{|q|\gamma_f} < \frac{\log 2}{\delta} < \frac{1}{p}$$

and

$$\sigma = \left(\frac{\log 2}{\delta} + \varepsilon\right) < \frac{1}{p} \qquad \text{for some } \varepsilon > 0.$$

Therefore

$$m_n = m(n) = o(e^{\sigma\delta n}).$$

This means that t^p is σ-strongly non-equivalent to $E_{F,1}$, and with an appeal to Proposition 1.3, we conclude the proof. q.e.d.

As an easy consequence we obtain a result of Lindenstrauss and Tzafriri ([9], [10] p. 163) proved by using the method of constructing Orlicz functions associated with 0–1 valued sequences:

Corollary 1.7. *For any $p > 1$ there exists a minimal reflexive Orlicz sequence space l^F with indices $\alpha_F = \beta_F = p$ which does not have any complemented copy of l^p.*

Proof. Just take $q = (4p\log 2)/\gamma_f$ and apply Theorem 1.6.

Corollary 1.8. *If $1 < p \neq 2$ and $q \neq 0$ satisfy*

$$\frac{p}{|q|} < \frac{\gamma_f}{2\log 2},$$

then the Orlicz function space $L^{F_{p,q}}$ does not contain any complemented copy of l^p.

Proof. If follows from Proposition 8 in ([3]) and Theorem 1.6.

Remark. An open question is to determine values $p \neq 2$ and q such that the Orlicz space $L^{F_{p,q}}$ contains a complemented copy of l^p. Any positive result in this direction would automatically imply that Problem 4.b.8 in ([10]) has a negative solution, i.e. the existence of minimal Orlicz sequence spaces which are not prime.

II. Sequence spaces containing singular l^p-complemented copies

In ([6] p. 276) N. Kalton proved the existence of Orlicz sequence spaces l^F, with different indices $\alpha_F \neq \beta_F$, containing l^p-complemented copies ($p > \frac{3}{2}$) such that the function t^p is not equivalent to any function in the set $E_{F,1}$. This means, in other words, that the complemented subspaces isomorphic to l^p in l^F can never be the span of any block basis with constant coefficients of the unit vector basis of l^F. (We shall often say, in short, that the space l^F has a *singular* l^p-complemented copy.)

The first result in this section extends Kalton's example to the case of Orlicz sequence spaces with the same indices:

THEOREM 2.1. *Let $p > 1$. There exists an Orlicz sequence space l^F, with indices $\alpha_F = \beta_F = p$, containing a complemented subspace isomorphic to l^p but t^p is not equivalent at 0 to any function in $E_{F,1}$.*

PROOF. Let us define a sequence of positive functions $(f_n)_1^\infty$ on $[0, \infty)$ in the following way: For each $n \in \mathbf{N}$, f_n is the function of period $P_n = 2^{2^{2n}}$ such that

$$f_n(t) = \begin{cases} 0, & \text{if } 0 \leqq t \leqq P_n - 4 \cdot 2^n; \\ \sum_{k=1}^{n} (1 - \cos(\pi t / 2^k)), & \text{if } P_n - 4 \cdot 2^n \leqq t \leqq P_n. \end{cases}$$

We consider also the sequence of periodic functions $(g_n)_1^\infty$ defined in ([6], p. 275) by

$$g_n(t) = \begin{cases} 0, & \text{if } 0 \leqq t \leqq P_n - 4 \cdot 2^n, \\ \frac{1}{2}(t - P_n) + 2 \cdot 2^n, & \text{if } P_n - 4 \cdot 2^n \leqq t \leqq P_n - 2 \cdot 2^n, \\ \frac{1}{2}(P_n - t), & \text{if } P_n - 2 \cdot 2^n \leqq t \leqq P_n, \end{cases}$$

and the functions

$$f(t) = \sup_{n \in \mathbf{N}} f_n(t) \quad \text{and} \quad g(t) = \sup_{n \in \mathbf{N}} g_n(t).$$

If $g(t) \neq 0$ let $n(t)$ be the biggest integer n for which $g_n(t) = g(t)$. Noticing that $f_n(t) < f_m(t)$ implies $g_n(t) \leqq g_m(t)$, it is easy to check that $f(t) = f_{n(t)}(t)$.

We define now the function

$$F(t) = t^p \exp\{qf(-\log t)\} \qquad \text{for } 0 < t < 1$$

with $0 < q < (p-1)/3\pi$. It is straightforward to show that F is a convex function.

Let us prove that t^p is not equivalent to any function in $E_{F,1}$. Suppose that this is not the case, i.e. there exists $\bar{F} \in E_{F,1}$ with $\bar{F} \sim t^p$ at 0. Thus for some sequence $(s_k) \nearrow \infty$,

$$\bar{F}(t) = \lim_{k\to\infty} \frac{F(e^{-s_k}t)}{F(e^{-s_k})} = t^p \exp\left\{q\left(\lim_{k\to\infty} (f(s_k - \log t) - f(s_k)\right)\right\}$$

$$= t^p e^{q\bar{f}(-\log t)}$$

for $t \in [0, 1]$, and where $\bar{f}$ means the function

$$\bar{f}(x) = \lim_{k\to\infty} (f(x + s_k) - f(s_k)).$$

Let $g(x + s_k) \neq 0$ and $n(x + s_k) = n_k(x) \equiv n_k > 1$; there are two points t'_k and t''_k with $t'_k \leqq x \leqq t''_k$ satisfying $t''_k - t'_k = 4 \cdot 2^{n_k}$ and $n(t + s_k) = n_k$ for every $t'_k < t < t''_k$. Let us denote by $(t'_x, t''_x) \in \bar{\mathbf{R}} \times \bar{\mathbf{R}}$ ($\bar{\mathbf{R}} \equiv \mathbf{R} \cup \{\pm\infty\}$) an accumulation point of the sequence $\{(t'_k, t''_k)\} \subset \mathbf{R}^2$. Thus $-\infty \leqq t'_x \leqq x \leqq t''_x \leqq \infty$ and $t''_x - t'_x = 4 \cdot 2^{m(x)}$ for some integer $m(x) \in \mathbf{N} \cup \{\infty\}$. We still denote by (t'_k, t''_k) the subsequence converging to (t'_x, t''_x) in $\bar{\mathbf{R}} \times \bar{\mathbf{R}}$. Furthermore we can assume that t'_x is finite or $t''_x = +\infty$. Indeed, if $t'_x = -\infty$ and $t''_x < \infty$ we could take $x' > t''_x$ getting that $t'_{x'} \geqq t''_x$, hence $t'_{x'}$ would be finite.

Let us now distinguish the cases $m(x) \equiv m$ finite or infinite. If $m = +\infty$, then $m(t) = \infty$ for $t'_x \leqq t < \infty$. Arguing as in the proof of Theorem 1.2 we obtain a scalar sequence (σ_i) such that $0 \leqq \sigma_i \leqq 2^{i+1}$ and $\lim_{k\to\infty} s_k = \sigma_i$ (mod 2^{i+1}). Thus, by Weierstrass criterion, for $t \geqq t'_x$

$$\lim_{k\to\infty}\left[f(t + s_k) - \sum_{i=1}^{n_k(t)}\left(1 - \cos\frac{\pi s_k}{2^i}\right)\right] = \lim_{k\to\infty}\sum_{i=1}^{n_k(t)}\left(\cos\frac{\pi s_k}{2^i} - \cos\frac{\pi(t + s_k)}{2^i}\right)$$

$$= \sum_{i=1}^{\infty}\left(\cos\frac{\pi\sigma_i}{2^i} - \cos\frac{\pi(t + \sigma_i)}{2^i}\right).$$

Hence, there exists

$$\lim_{k\to\infty}\left[f(s_k)-\sum_{i=1}^{n_k}\left(1-\cos\frac{\pi s_k}{2^i}\right)\right]$$

$$=\lim_{k\to\infty}\left(f(s_k)-f(t+s_k)+f(t+s_k)-\sum_{i=1}^{n_k(t)}\left(1-\cos\frac{\pi s_k}{2^i}\right)\right]$$

$$=\sum_{i=1}^{\infty}\left(\cos\frac{\pi\sigma_i}{2^i}-\cos\frac{\pi(t+\sigma_i)}{2^i}\right)-\bar{f}(t)=C,$$

where C is a constant (not depending on t). Therefore, the expression

$$\bar{f}(t)=\sum_{i=1}^{\infty}\left(\cos\frac{\pi\sigma_i}{2^i}-\cos\frac{\pi(t+\sigma_i)}{2^i}\right)-C$$

results for $t\geqq t_x'$, so the function $\bar{f}$ is not bounded.

Let us see now that also in the case of $m(x)\equiv m$ finite, the function $\bar{f}$ is unbounded. From the uniform convergence on compact subsets and from the existence of $\lim_{k\to\infty}f(s_k)=C'$, we obtain

$$\bar{f}(t)=\lim_{k\to\infty}\,[f(t+s_k)-f(s_k)]=\sum_{i=1}^{m}\left(1-\cos\frac{\pi(t+\sigma_i)}{2^i}\right)-C'$$

for every $t_x'\leqq t\leqq t_x''$. Then, as in Lemma 1.5, the oscillation of $\bar{f}$ in the interval $[t_x',t_x'']$ of large $4\cdot 2^{m(x)}$ is bigger than or equal to $C_0m(x)$ for a constant $C_0>0$. Hence, $\bar{f}$ bounded implies $m(x)$ is bounded. If

$$\bar{g}(t)=\lim_{k\to\infty}\,[g(t+s_k)-g(s_k)],$$

since

$$0\leqq g(t+s_k)=g_{n(t+s_k)}(t+s_k)\leqq 2^{n(t+s_k)}\to 2^{m(t)}$$

for $k\to\infty$, we have that $(g(s_k))_1^\infty$ is a bounded sequence, and so the function $\bar{g}$ is bounded. This is a contradiction, because if $G(t)=t^2\exp[g(-\log t)]$, by Kalton ([6] p. 277), there is no function

$$\bar{G}(t)=t^2e^{\bar{g}(-\log t)}$$

in $E_{G,1}$ equivalent to t^2 at 0.

In both cases we conclude that $\bar{f}$ is unbounded, so the function t^p is not equivalent to any function $\bar{F}(t)=t^p\exp\{q\bar{f}(-\log t)\}$ in $E_{F,1}$.

To prove that l^F contains a complemented copy of l^p we show that the inclusion map from l^F into l^p is not a strictly singular operator and we apply

Theorem 5.4 of [6]. It is clear that the inclusion map $J: l^F \to l^p$ is bounded. To show that J is not strictly singular, we use the analytic criterion given in ([6] Thm. 5.3). Put $\varepsilon_n = e^{-P_n}$, $n \in \mathbf{N}$. Then

$$\frac{1}{\log(1/\varepsilon_n)} \int_{\varepsilon_n}^{1} \frac{F(t)}{t^{p+1}} dt = \frac{1}{P_n} \int_{\varepsilon_n}^{1} \frac{e^{qf(-\log t)}}{t} dt = \frac{1}{P_n} \int_0^{P_n} e^{qf(u)} du.$$

Notice that, for $t \in [0, P_n]$, $f(t) = \max\{f_i(t) : 1 \leqq i \leqq n\}$. Hence, by using the definition of f_i

$$\begin{aligned} \frac{1}{P_n} \int_0^{P_n} e^{qf(u)} du &= 1 + \frac{1}{P_n} \int_0^{P_n} (e^{qf(u)} - 1) du \\ &\leqq 1 + \frac{1}{P_n} \sum_{i=1}^{n} \int_0^{P_i} (e^{qf_i(u)} - 1) du \\ &= 1 + \sum_{i=1}^{n} \frac{1}{P_i} \int_0^{P_i} (e^{qf_i(u)} - 1) du \\ &\leqq 1 + \sum_{i=1}^{n} \frac{1}{P_i} \left(\int_{P_i - 4 \cdot 2^i}^{P_i} e^{qf_i(u)} du \right) \\ &\leqq 1 + \sum_{i=1}^{n} \frac{1}{P_i} e^{2qi} 4 \cdot 2^i \\ &\leqq 1 + 4 \sum_{i=1}^{\infty} 2^{-2^{2i}} e^{2qi} 2^i < \infty \end{aligned}$$

for every integer n, and this proves that J is not strictly singular.

Finally, it remains to show that $\alpha_F = \beta_F = p$. Since

$$\frac{F(\lambda t)}{F(\lambda) t^p} = e^{q[f(-\log \lambda t) - f(-\log \lambda)]},$$

we need to prove that for every $\varepsilon > 0$ there is a constant $K_\varepsilon > 0$ such that

$$|f(x+t) - f(x)| < \varepsilon t + K_\varepsilon \qquad \text{for } t \geqq 0.$$

For this let us show the inequality

$$|f(x+t) - f(x)| \leqq 2 \log_2 t + B \qquad \text{for } t \geqq 1$$

with $B = \pi + 2$.

Firstly suppose that in $(x, x+t)$ there is no zero of f. Then for a natural number m there holds

$$f(x+t)=\sum_{k=1}^{m}\left(1-\cos\frac{\pi(x+t)}{2^k}\right) \quad \text{and} \quad f(x)=\sum_{k=1}^{m}\left(1-\cos\frac{\pi x}{2^k}\right).$$

Hence,

$$|f(x+t)-f(x)| \leqq \sum_{k=1}^{m}\left|\cos\frac{\pi x}{2^k}-\cos\frac{\pi(x+t)}{2^k}\right|$$

and for $2^{n-1} \leqq t < 2^n$ we get that

$$|f(x+t)-f(x)| \leqq 2n+\sum_{k=n+1}^{\infty}\left|\cos\frac{\pi x}{2^k}-\cos\frac{\pi(x+t)}{2^k}\right| \leqq 2n+\sum_{k=n+1}^{\infty}\frac{\pi t}{2^k}$$

$$\leqq 2(\log_2 t+1)+\pi = 2\log_2 t + B.$$

Suppose now that the interval $(x, x+t)$ contains zeros of the function f, and let x_0 and x_1 be the first and the last zeros, respectively. Then, by above inequality we have

$$f(x+t)=|f(x+t)-f(x_1)| \leqq 2\log_2 t + B$$

and

$$f(x)=|f(x_0)-f(x)| \leqq 2\log_2 t + B.$$

Hence, as $f \geqq 0$, we conclude that

$$|f(x+t)-f(x)| \leqq 2\log_2 t + B. \qquad \text{q.e.d.}$$

Remark 1. The complementary function $\hat{F}$ of the above function F verifies also that the Orlicz sequence space $l^{\hat{F}}$ contains a *singular* l^r-complemented copy ($1/r+1/p=1$). This follows from ([10] Thm. 4.b.3).

Remark 2. In the case of $p=1$ the above theorem does not hold: Every Orlicz sequence space l^F with $\alpha_F=1$ contains a complemented subspace isomorphic to l^1 generated by a block basis with constant coefficients of the unit vector basis of l^F (cf. [7] Thm. 4.2 or [2] Prop. 10).

We use Theorem 2.1 to study the "inverse problem" for *singular* l^p-complemented copies: given an arbitrary set H of real numbers $p>1$, find an Orlicz sequence space l^F containing only singular l^p-complemented copies and such that the set of values p for which l^p is isomorphic to a complemented subspace of l^F is exactly the set H.

THEOREM 2.2. *Let H be a closed set of positive numbers with $1 < \alpha = \inf H \leqq \sup H = \beta < \infty$. Then there exists an Orlicz sequence space l^F with indices $\alpha_F = \alpha$ and $\beta_F = \beta$ which contains complemented subspaces isomorphic to l^p if and only if $p \in H$. Furthermore, for each $p \in H$, t^p is not equivalent to any function in $E_{F,1}$.*

PROOF. First, we pick up a dense sequence $(p_n)_1^\infty$ in the set H, so that every element of the range of (p_n) appears infinitely many times in the sequence.

Fix $0 < q < (\alpha - 1)/3\pi$; we consider the function defined in the above theorem,

$$F_p(t) = t^p e^{qf(-\log t)}, \qquad \text{for } 0 < t < 1,$$

which satisfies $\alpha_{F_p} = \beta_{F_p} = p$, l^{F_p} contains a l^p-complemented copy, and t^p is not equivalent to any function in $E_{F_p,1}$.

Define a continuous function F on $[0, 1]$ by $F(0) = 0$, $F(1) = 1$ and

$$F(x) = F_{p_n}\left(\frac{x}{r_n}\right) F(r_n) \qquad \text{for } r_{n+1} \leqq x \leqq r_n,$$

where $r_n = e^{-n^2}$. This function F is not necessarily convex, but as $F(x)/x$ is an increasing function, we get that F is equivalent to a convex function at 0.

Since

$$\frac{F(r_n x)}{F(r_n)} = F_{p_n}(x) \qquad \text{for } 1 \geqq x \geqq \frac{r_{n+1}}{r_n}$$

and $r_{n+1}/r_n \to 0$ if $n \to \infty$, we get that $F_{p_n} \in E_{F,1}$ for all $n \in \mathbf{N}$, and hence $F_p \in E_{F,1}$ for each $p \in H$. Now, by ([10] p. 150), l^F has a complemented copy of l^{F_p}, so, recalling Theorem 2.1, we deduce that l^F has a l^p-complemented copy for each $p \in H$.

Let us see that t^p is not equivalent to any function in $E_{F,1}$. Assuming the opposite there exists a function $G \in E_F$, which is equivalent to t^p at 0. Thus for a sequence $(s_k) \nearrow \infty$ we have

$$G(x) = \lim_{k \to \infty} \frac{F(e^{-s_k} x)}{F(e^{-s_k})}$$

uniformly on $[0, 1]$. Fix $0 < x < 1$; let (n_k) be an integer sequence satisfying

$$r_{n_k} = e^{-n_k^2} \geqq e^{-s_k} x \geqq e^{-(n_k+1)^2} = r_{n_k+1}.$$

Hence

$$(n_k^2 - s_k) \leqq \log \frac{1}{x} \leqq (n_k + 1)^2 - s_k.$$

Let $(t'_x, t''_x) \in \bar{\mathbf{R}} \times \bar{\mathbf{R}}$ be an accumulation point of the sequence $\{(n_k^2 - s_k, (n_k + 1)^2 - s_k)\}_{k=1}^{\infty} \subset \mathbf{R}^2$, so $t''_x - t'_x = \infty$. Moreover we can assume, w.l.o.g., that $t''_x = \infty$. Indeed, if $t''_x < \infty$ we can take $0 < x' < x$ with $\log(1/x') \geqq t''_x$, getting $t''_x \leqq t'_{x'} < \infty$, and hence $t''_{x'} = \infty$.

Taking a sufficiently small scalar t_0 with $\log(1/t_0) \geqq t'_x$, and by passing to a subsequence if necessary, we get, for $0 < t < t_0$,

$$r_{n_k} \geqq e^{-s_k} t \geqq r_{n_k+1},$$

so

$$F(e^{-s_k}t) = \left(\frac{e^{-s_k}t}{r_{n_k}}\right)^{p_{n_k}} F(r_{n_k}) e^{qf(s_k - n_k^2 - \log t)}.$$

At this stage, we continue in a similar way as in the above theorem. Thus, by passing to a subsequence if necessary, there exist a $p_0 \in H$ and a constant $K > 0$ such that

$$G(t) = Kt^{p_0} \exp\left\{q \lim_{k \to \infty} [f(s_k - n_k^2 - \log t) - f(s_k - n_k^2)]\right\}$$

for $0 < t < t_0$. Therefore, the function G is equivalent at 0 to a function of $E_{F_{p_0},1}$ but this is impossible because G is equivalent to t^p and from Theorem 2.1 we know that $t^p \notin E_{F_{p_0},1}$ for any $p > 1$.

Now we show that t^p is strongly non-equivalent to $E_{F,1}$ for each $p \notin H$, which implies, by ([10] Theorem 4.b.5), that l^F does not contain any complemented subspace isomorphic to l^p. Fix $p \notin H$; let $\varepsilon > 0$ be such that $(p - 3\varepsilon, p + 3\varepsilon) \cap H = \emptyset$. For each $n \in \mathbf{N}$, put $m(n) = n^2$ and assume the existence of an integer k so that

$$(*) \qquad e^{-\varepsilon n} \leqq \frac{F(\tau^k \tau^i)}{F(\tau^k)\tau^{ip}} \leqq e^{\varepsilon n}$$

for $i = 1, 2, \ldots, n^2$ and $\tau = e^{-1}$. Let $1 \leqq j \leqq n^2 - n$ $(n > 1)$; by the above inequality with $i = j$ and $i = j + n$,

$$(**) \qquad e^{-2\varepsilon n} \tau^{pn} \leqq \frac{F(\tau^{k+j+n})}{F(\tau^{k+j})} \leqq e^{2\varepsilon n} \tau^{pn}$$

for $1 \leqq j \leqq (n^2 - n)$.

Depending on the possible values of k, we consider a particular value of j as follows:

(a) $j=(n-1)^2-k$ when $k<(n-1)^2$,

(b) $j=1$ when $(m-1)^2 \leqq k<(k+n)<m^2$ for an integer $m \geqq n$,

(c) $j=m^2-k$ when $(m-1)^2 \leqq k<m^2 \leqq k+n$ for an integer $m \geqq n$.

It is easy to check that in every case

$$\frac{F(\tau^{k+j+n})}{F(\tau^{k+j})}=\tau^{p_r n} e^{q(f(\sigma+n)-f(\sigma))}$$

for some $p_r \in H$ and $0 \leqq \sigma \leqq k+j$. By the inequality (**),

$$\frac{F(\tau^{k+j+n})}{F(\tau^{k+j})\tau^{pn}} \notin [e^{-\varepsilon n}, e^{\varepsilon n}]$$

for n big enough ($n \geqq n_0$). Indeed, assuming the opposite, if n_0 denotes an integer such that

$$q\cdot(2\log_2 n+B)<\varepsilon n$$

for $n \geqq n_0$ and $B=\pi+2$, and making use of the inequality given in Theorem 2.1

$$|f(\sigma+n)-f(\sigma)| \leqq 2\log_2 n+B$$

for $\sigma \geqq 0$, we easily find that $|p_r-p|<2\varepsilon$, which is not possible.

Thus, we have arrived at a contradiction with (*) and so we conclude that for any integer $n \geqq n_0$ there exist $m(n)=n^2$ points in (0, 1) such that for any integer k there exists at least one index $i=1,2,\ldots,n^2$ for which

$$\frac{F(\tau^k\tau^i)}{F(\tau^k)\tau^{p_i}} \notin [e^{-\varepsilon n}, e^{\varepsilon n}].$$

As $m(n+1)=o(e^{\sigma\varepsilon n})$ for any $\sigma>0$, and by the Δ_2-condition, t^p is strongly non-equivalent to $E_{F,1}$.

Finally, it remains to show that $\alpha_F=\alpha$ and $\beta_F=\beta$. Let $p=\alpha-\varepsilon$ with $\varepsilon>0$; for $r_n \geqq \lambda>\lambda t \geqq r_{n+1}$ we have

$$\frac{F(\lambda t)}{F(\lambda)t^p}=t^{p_n-p}\exp\{q[f(-\log(\lambda t)+n^2)-f(-\log\lambda+n^2)]\}$$

$$\leqq t^{\varepsilon}\exp\{q[2\log_2(-\log t)+B]\}$$

for $p_n \in H$. Then there exists a constant $K > 0$ and a sufficiently small scalar $t_0 < 1$ such that

$$\frac{F(\lambda t)}{F(\lambda)t^p} < 1 \qquad \text{if } 0 < t < t_0$$

and

$$\frac{F(\lambda t)}{F(\lambda)t^p} \leqq K \qquad \text{if } t_0 \leqq t \leqq 1.$$

In the general case, as

$$\frac{F(\lambda t)}{F(\lambda)t^p} = \frac{F(\lambda t)}{F(r_n)\left(\frac{\lambda t}{r_n}\right)^p} \frac{F(r_n)}{F(r_{n-1})\left(\frac{r_n}{r_{n-1}}\right)^p} \cdot \cdots \cdot \frac{F(r_{n-k})}{F(\lambda)\left(\frac{r_{n-k}}{\lambda}\right)^p}$$

for $r_{n+1} \leqq \lambda t \leqq r_n \leqq \cdots \leqq r_{n-k} \leqq \lambda \leqq r_{n-k-1}$, we obtain that

$$\frac{F(\lambda t)}{F(\lambda)t^p} \leqq K^{m_0+2}$$

for every $0 \leqq t$, $\lambda \leqq 1$, where m_0 is the number of integers m satisfying $(r_m/r_{m-1}) \geqq t_0$. Hence $p \leqq \alpha_F$ for any $p < \alpha$, so $\alpha \leqq \alpha_F$ and, as $l^F \supsetneq l^\alpha$, we conclude that $\alpha_F = \alpha$.

The proof of the remaining equality $\beta_F = \beta$ is similar. q.e.d.

In the general case of arbitrary closed sets H the "inverse problem" for *singular* l^p-complemented copies has also a positive solution:

THEOREM 2.3. *Let $1 < \alpha \leqq \beta < \infty$ and H be an arbitrary closed subset of the interval $[\alpha, \beta]$. Then there exists an Orlicz sequence space l^F with indices $\alpha_F = \alpha$ and $\beta_F = \beta$, which contains a complemented copy of l^p if and only if $p \in H$. Furthermore t^p is not equivalent to any function in $E_{F,1}$ for any $p \geqq 1$.*

PROOF. We proceed in a similar way as in the proof of Theorem 2 of ([3]), which makes use of the method of constructing Orlicz functions F_p associated with sequences of 0's and 1's given in ([9], [10] p. 161). Thus we shall only give a sketch of the proof.

We can assume that $H \neq \emptyset$ and let $(p_{2n-1})_{n=1}^\infty$ be a dense sequence in the set H, so that every element of the range of the sequence appears infinitely many times. Fixed $0 < q < (\alpha - 1)/3\pi$, we consider the Orlicz functions

$$F_{p_n}(t) = t^{p_n} \exp\{qf(-\log t)\}$$

for n odd and where f is the function defined in Theorem 2.1.

Let (m_n) be the integer sequence considered in Theorem 2 of ([3]) to construct the sequence $\rho = (\rho(n))_1^\infty$ of 0's and 1's, and let F_ρ denote its associated Orlicz function with indices $\alpha = \alpha_F$ and $\beta = \beta_F$. Define the continuous function F on [0, 1] as follows: $F(0) = 0$, $F(1) = 1$, and

$$F(t) = \begin{cases} F_\rho\left(\dfrac{t}{r_n}\right) F(r_n) & \text{if } r_{n+1} \leqq t \leqq r_n \text{ and } n \text{ even} \\[2ex] F_{p_n}\left(\dfrac{t}{r_n}\right) F(r_n) & \text{if } r_{n+1} \leqq t \leqq r_n \text{ and } n \text{ odd} \end{cases}$$

where $r_n = e^{-m_n}$ for $n \in \mathbf{N}$.

This function F is equivalent at 0 to a convex function (f.i. $F_0(x) = \int_0^x (F(t)/t)dt$), since $F(t)/t$ is an increasing function.

In the same way as in Theorem 2.2, by considering the sequence $(F(r_n x)/F(r_n))$ for n odd, we get that the function F_p belongs to E_F for each $p \in H$. Thus, using ([10] p. 150) and Theorem 2.1 we deduce that the Orlicz sequence space l^F has a complemented copy of l^p for each $p \in H$.

Let us see that t^p is not equivalent to any functions in $E_{F,1}$. Suppose the opposite, so there exists a function $G \in E_F$ which is equivalent to t^p at 0, and a scalar sequence $(s_k)_1^\infty \searrow 0$ such that

$$G(t) = \lim_{k \to \infty} \frac{F(s_k t)}{F(s_k)}$$

uniformly in [0, 1]. Now, reasoning as in Theorem 2.2, there exist $t_0 > 0$ such that for $0 < t < t_0$ we have (passing to a subsequence if necessary)

$$G(t) = \lim_{k \to \infty} \frac{F_\rho\left(\dfrac{s_k t}{r'_k}\right) F(r'_k)}{F(s_k)} \quad \text{or} \quad G(t) = \lim_{k \to \infty} \frac{F_{p'_k}\left(\dfrac{s_k t}{r'_k}\right) F(r'_k)}{F(s_k)}$$

for a subsequence (p'_k) of the sequence (p_n). Thus if $p \in H$ is an accumulation point of the sequence (p'_k), we have

$$G(t) = \lim_{k \to \infty} \frac{F_p\left(\dfrac{s_k t}{r'_k}\right)}{F_p\left(\dfrac{s_k}{r'_k}\right)} \lim_{k \to \infty} \frac{F_p\left(\dfrac{s_k}{r'_k}\right) F(r'_k)}{F(s_k)}$$

or

$$G(t)=\lim_{k\to\infty}\frac{F_p\left(\dfrac{s_k t}{r'_k}\right)}{F_p\left(\dfrac{s_k}{r'_k}\right)}\lim_{k\to\infty}\frac{F_{p_k}\left(\dfrac{s_k}{r'_k}\right)F(r'_k)}{F(s_k)}.$$

This implies that t^p is equivalent at 0 to a function of $E_{F_p,1}$ or $E_{F_p,1}$, which is a contradiction.

Finally, following the method developed in ([3] Theorem 2) and the tricks of Theorem 2.1, it can be shown that for each $p \notin H$ the function t^p is strongly non-equivalent to $E_{F,1}$ and that the associated indices satisfy $\alpha_F=\alpha$ and $\beta_F=\beta$. Hence l^F does not contain any complemented subspace isomorphic to l^p for $p \notin H$. q.e.d.

III. Function spaces containing singular l^p-complemented copies

We start this section introducing the following

DEFINITION. Given a Banach lattice X and a Banach space Y, an operator $T: X \to Y$ is said to be *disjointly singular* if there is no disjoint sequence of non-null vectors (u_n) in X such that the restriction of the operator T to the subspace $[u_n]$ spanned by the vectors (u_n) is an isomorphism. ("Disjoint sequence" means that $|u_n| \wedge |u_m| = 0$ for $n \neq m$.)

Recall that an operator $T: X \to Y$ is *strictly singular* if it fails to be an isomorphism on any infinite-dimensional subspace. Clearly every strictly singular operator is a disjointly singular operator. However the converse is not true as the following easy example shows:

Let us consider the canonic inclusion map $J: L^q(0,1) \to L^p(0,1)$ for $1 \leqq p < q$. This operator J is not strictly singular, since its restriction to the subspace $[r_n]_q$ spanned by the Rademacher functions (r_n) in $L^q(0,1)$ is an isomorphism ($[r_n]_q \approx [J(r_n)]_p \approx l^2$). However, the operator J is disjointly singular because for any sequene of non-null functions (f_n) in $L^q(0,1)$ with pairwise disjoint support, we have that

$$[f_n]_q \approx \left[\frac{f_n}{\|f_n\|_q}\right]_q \approx l^q \quad\text{and}\quad [J(f_n)]_p \approx \left[\frac{f_n}{\|f_n\|_p}\right] \approx l^p$$

(cf. [5] Lemma 1).

For operators defined on separable Orlicz sequence spaces it is true that to be disjointly singular is the same as to be strictly singular. This follows directly from a basic result on bases (cf. [10], Prop. 1.a.11).

We study now when the inclusion map $J: L^F(0, 1) \to L^G(0, 1)$ between separable Orlicz function spaces is a disjointly singular operator. (Note that the inclusion operator J cannot be strictly singular ever.) We wish to find an analytic criterion for the inclusion map $J: L^F(0, 1) \to L^p(0, 1)$ to be disjointly singular. We will use basically the method used in [6] for the strict singularity in Orlicz sequence spaces.

Let F and G be Orlicz functions verifying the Δ_2-condition at ∞ and $L^F(0, 1) \equiv L^F \subset L^G(0, 1) \equiv L^G$. Thus, the function

$$W(t) = \frac{G(t)}{F(t)} \qquad \text{for } t \geqq 1$$

is bounded on $I_\infty = [1, +\infty)$. Let us denote by $W(t)$ its unique extension to the Stone–Čech compactification βI_∞ of I_∞. Similarly we shall denote by $F_\tau(x)$ the extension to βI_∞ of the functions F_τ defined by $F_\tau(x) = F(\tau x)/F(\tau)$, for $\tau \in I_\infty$.

THEOREM 3.1. *Suppose $L^F \subset L^G$. Then the inclusion map $J: L^F \to L^G$ is not a disjointly singular operator if and only if there exists a constant $C > 0$ and a probability measure μ on βI_∞ such that $\mu(I_\infty) = 0$ and*

$$\int F_\tau(x) d\mu(\tau) \leqq C \int W(\tau) G_\tau(x) d\mu(\tau)$$

for $0 \leqq x \leqq 1$.

PROOF. Assume that the map J is not disjointly singular. So there exists a basic sequence of functions (u_n) of norm one in L^F with mutually disjoint supports such that the restriction of J to the subspace $[u_n]$ is an isomorphism. Reasoning as in Proposition 4.3 of ([9]) and passing to a subsequence, if necessary, we get two functions $H_1 \in C_F^\infty$ and $H_2 \in C_G^\infty$ verifying that

$$\left| \, |xu_n|_F - H_1(x) \right| \leqq 1/2^n \quad \text{and} \quad \left| \, |xu_n|_G - H_2(x) \right| \leqq 1/2^n$$

for every $0 \leqq x \leqq 1$ and $n \in \mathbf{N}$.

Thus the series $\Sigma \lambda_n u_n$ converges in L^F if and only if $\Sigma H_1(|\lambda_n|) < \infty$, and in L^G if and only if $\Sigma H_2(|\lambda_n|) < \infty$. Hence, H_1 and H_2 are equivalent functions at 0. So we have $H_1(x) \leqq CH_2(x)$ for $0 < x \leqq 1$ and a constant $C > 0$.

Now, as we can assume that the functions (u_n) are simple with

$$\inf\{|u_n(t)| : t \in \operatorname{supp} u_n\} \to \infty \qquad \text{for } n \to \infty,$$

there exists a probability measure μ_n with support in I_∞ such that

$$|xu_n|_F = \int_{\beta I_\infty} F_\tau(x)d\mu_n(\tau) \qquad (0 < x \leqq 1)$$

and

$$|xu_n|_G = \int_{\beta I_\infty} W(\tau)G_\tau(x)d\mu_n(\tau).$$

Then, it results that

$$H_1(x) = \int_{\beta I_\infty} F_\tau(x)d\mu(\tau) \quad \text{and} \quad H_2(x) = \int_{\beta I_\infty} W(\tau)G_\tau(x)d\mu(\tau)$$

for μ a probability measure, which is a weak* accumulation point of the sequence (μ_n), so $\mu(I_\infty) = 0$. Thus the proof of the necessity implication is finished.

Let us see the sufficiency implication. We can suppose w.l.o.g. that $F(s) \geqq G(s)$ for every $s > 0$, so we assume that there exists a probability measure μ on βI_∞ with $\mu(I_\infty) = 0$ verifying that

$$(*) \qquad \int W(\tau)G_\tau(x)d\mu(\tau) \leqq \int F_\tau(x)d\mu(\tau) \leqq C \int W(\tau)G_\tau(x)d\mu(\tau)$$

for $0 \leqq x \leqq 1$.

For each $n \in \mathbf{N}$ there exists a probability measure $\mu_n = \sum_{n=1}^{m_n} \alpha_{n,i}\delta_{t_{n,i}}$ on I_∞ with $t_{n,i} \in [n, +\infty)$ such that

$$\left| \int F_\tau(x)d\mu(\tau) - \int F_\tau(x)d\mu_n(\tau) \right| \leqq \frac{1}{2^n}$$

and

$$\left| \int W(\tau)G_\tau(x)d\mu(\tau) - \int W(\tau)G_\tau(x)d\mu_n(\tau) \right| \leqq \frac{1}{2^n}$$

for $0 \leqq x \leqq 1$. Thus, for $0 < \lambda_n < 1$, using $(*)$ we obtain

$$\sum_{n=1}^{\infty} \int \frac{F(t|\lambda_n|)}{F(t)} d\mu_n(t) = \sum_{n=1}^{\infty} \int F_t(|\lambda_n|)d\mu_n(t)$$

converges if and only if

$$\sum_{n=1}^{\infty} \int W(t)\frac{G(t|\lambda_n|)}{G(t)} d\mu_n(t) = \sum_{n=1}^{\infty} \int W(t)G_t(|\lambda_n|)d\mu_n(t)$$

converges.

Since $\operatorname{supp}\mu_n \subset [n, +\infty)$, we can assume, by passing to a subsequence, if necessary, that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{m_n}\frac{\alpha_{n,i}}{F(t_{n,i})}=\sum_{n=1}^{\infty}\int\frac{d\mu_n(t)}{F(t)}\leqq 1.$$

Let $(A_{n,i})$ be a sequence of pairwise disjoint intervals in $[0, 1]$ with length $\bar{\mu}(A_{n,i}) = \alpha_{n,i}/F(t_{n,i})$. Then, with the functions $u_n = \sum_{i=1}^{m_n} t_{n,i}\chi_{A_{n,i}}$, we get

$$\int\frac{F(xt)}{F(t)}\,d\mu_n(t)=\sum_{i=1}^{m_n}\frac{F(xt_{n,i})}{F(t_{n,i})}\alpha_{n,i}=\sum_{i=1}^{m_n}F(xt_{n,i})\bar{\mu}(A_{n,i})=\int_0^1 F(xu_n)dt$$

and, similarly, for the function G, i.e.

$$\int W(t)\frac{G(xt)}{G(t)}\,d\mu_n(t)=\int_0^1 G(xu_n)dt.$$

Thus, the series $\sum \lambda_n u_n$ converges in L^F if and only if it converges in L^G. Hence, the restriction of the inclusion map J to the subspace $[u_n]$ is an isomorphism, and so the inclusion map $J: L^F \to L^G$ is not a disjointly singular operator.

q.e.d.

Reasoning in a similar way as in ([6] Theorem 5.2) there results:

PROPOSITION 3.2. *Suppose $L^F \subset L^G$. The following conditions are equivalent*:

(a) *The inclusion map $J: L^F \to L^G$ is disjointly singular.*

(b *For any $C > 0$, there exist distinct points $x_1, x_2, \ldots, x_n \in I_\infty$ and $c_1, \ldots, c_n > 0$ such that*

$$\sum_{i=1}^{n} c_i F(tx_i) \geqq C \sum_{i=1}^{n} c_i G(tx_i) \qquad (t \geqq 1).$$

(c) *For any $C > 0$ there exists $a > 1$ and a positive Borel measure μ with support contained in $[1, a]$ such that*

$$\int F(tx)d\mu(x) \geqq C \int G(tx)d\mu(x) \qquad (t \geqq 1).$$

PROPOSITION 3.3. *Suppose $L^F \subset L^p$ $(p \geqq 1)$. Then the inclusion map $J: L^F \to L^p$ is disjointly singular if and only if*

$$(+) \qquad \liminf_{a\to\infty}\,\inf_{s\geqq 1}\frac{1}{\log a}\int_1^a\frac{F(su)}{s^p u^{p+1}}\,du=\infty.$$

PROOF. From (+) it follows that for any $C > 0$ there exists $a > 1$ such that

$$\int_1^a \frac{F(su)}{s^p u^{p+1}} du \geqq C \log a = C \int_1^a \frac{du}{u} \qquad (s \geqq 1).$$

So

$$\int_1^a \frac{F(su)}{u^{p+1}} du \geqq C \int_1^a \frac{(su)^p}{u^{p+1}} du,$$

and, by Proposition 3.2(c), we get that J is disjointly singular.

Suppose now that $J: L^F \to L^p$ is disjointly singular. Then, by Proposition 3.2, for any $C > 0$ there exists $1 \leqq x_1 < \cdots < x_n$ and $c_1, c_2, \ldots, c_n > 0$ such that

$$\sum_{i=1}^n c_i F(stx_i) \geqq C \sum_{i=1}^n c_i (tsx_i)^p \qquad (1 \leqq s, t).$$

For $a \geqq x_n^2$,

$$\int_1^{\sqrt{a}} \sum_{i=1}^n c_i \frac{F(stx_i)}{t^{p+1}} dt \geqq C \int_1^{\sqrt{a}} \sum_{i=1}^n c_i s^p x_i^p \frac{dt}{t}$$

$$= \frac{C}{2} \left(\sum_{i=1}^n c_i s^p x_i^p \right) \log a$$

and

$$\int_1^{\sqrt{a}} \sum_{i=1}^n c_i \frac{F(stx_i)}{t^{p+1}} dt = \sum_{i=1}^n c_i x_i^p \int_{x_i}^{\sqrt{a} x_i} \frac{F(su)}{u^{p+1}} du$$

$$\leqq \left(\sum_{i=1}^n c_i x_i^p \right) \int_1^a \frac{F(su)}{u^{p+1}} du.$$

Then, for any $C > 0$ there exists $x_n > 1$ such that, for any $a \geqq x_n^2$,

$$\frac{1}{\log a} \int_1^a \frac{F(su)}{s^p u^{p+1}} du \geqq \frac{C}{2}$$

for $s \geqq 1$. q.e.d.

COROLLARY 3.4. *Suppose $L^F \subset L^P$ ($p \geqq 1$) and L^F has not a complemented subspace generated by a sequence of non-null functions with pairwise disjoint support that is isomorphic to l^p. Then the condition (+) is satisfied.*

PROOF. If (+) is not satisfied, the inclusion map $J: L^F \to L^P$ is an isomorphism on some infinite-dimensional closed subspace X generated by a normalized sequence of functions (u_n) with pairwise disjoint support. Now, by ([5] Theorem 2), the span of the functions $u_n/\| u_n \|_p \equiv f_n$, $n \in \mathbf{N}$, in L^p is a complemented subspace isomorphic to l^p with projection $P: L^p \to J(X) = [f_n]_p$ defined by

$$P(h) = \sum_{n=1}^{\infty} \left(\int_{A_n} hg_n \right) f_n$$

where $A_n = \operatorname{supp} f_n$ and (g_n) is an orthonormal sequence to (f_n) in L^q $(1/q + 1/p = 1)$. Hence, considering the composition operator $J^{-1}PJ$, we get a projection of L^F onto X, which gives a contradiction. q.e.d.

We apply now the above result to prove the existence of Orlicz function spaces L^F containing *singular l^p-complemented copies*, i.e. Orlicz function spaces L^F having complemented subspaces isomorphic to l^p but the function t^p is not equivalent at 0 to any function in $E^{\infty}_{F,1}$. This is equivalent to saying that any complemented subspace isomorphic to l^p in L^F cannot be the span of any sequence of pairwise disjoint characteristic functions (χ_{A_n}) (cf. [2]).

The next theorem gives the extension to function spaces of a result of Kalton ([6] Example 2) and Theorem 2.1 stated in Orlicz sequence spaces:

THEOREM 3.5. *Let $p > 1$. There exists an Orlicz function space $L^F(0, 1)$, with indices $\alpha^{\infty}_F = \beta^{\infty}_F = p$, containing a complemented copy of l^p such that t^p is not equivalent at 0 to any function in $E^{\infty}_{F,1}$.*

PROOF. Let F be the function

$$F(t) = t^p \exp\{qf(\log t)\} \qquad \text{if } t > 1,$$

where $0 < q < (p-1)/3\pi$ and f is the function defined in Theorem 2.1.

Let us see that t^p is not equivalent to any function in $E^{\infty}_{F,1}$. If $\bar{F} \in E^{\infty}_{F,1}$ there exists a sequence $(s_k) \nearrow \infty$ such that

$$\bar{F}(t) = \lim_{k \to \infty} \frac{F(e^{s_k}t)}{F(e^{s_k})} = t^p \lim_{k \to \infty} e^{q(f(s_k + \log t) - f(s_k))}$$

$$= t^p e^{q\bar{f}(\log t)},$$

for $0 < t < 1$, and where the function $\bar{f}$ is defined by

$$\bar{f}(x) = \lim_{k \to \infty} [f(x + s_k) - f(s_k)].$$

As it was shown in Theorem 2.1 that $\bar{f}$ is not a bounded function, we get that $\bar{F}$ is not equivalent to t^p at 0.

We have $L^F \subset L^p$ since $\overline{\lim}_{t\to\infty}(t^p/F(t)) < \infty$, and the inclusion map $J: L^F \to L^p$ is not disjointly singular. Indeed, as

$$\int_1^a \frac{F(u)}{u^{p+1}}\,du = \int_1^a \frac{e^{qf(\log u)}}{u}\,du = \int_{1/a}^1 \frac{e^{qf(-\log t)}}{t}\,dt,$$

we deduce from Theorem 2.1 that

$$\lim_{a\to\infty}\inf_{s\geqq 1}\frac{1}{\log a}\int_1^a \frac{F(su)}{s^p u^{p+1}}\,du < \infty.$$

Thus, using Corollary 3.4, we conclude that the space L^F contains a complemented copy of l^p.

Finally it can be shown as in Theorem 2.1 that $\alpha_F^\infty = \beta_F^\infty = p$. q.e.d.

REMARK. In the case $p = 1$ the above result does not hold: Every Orlicz function space $L^F(0, 1)$ with $\alpha_F^\infty = 1$ contains a complemented subspace isomorphic to l^1 generated by a sequence of pairwise disjoint characteristic functions (cf. [2] Proposition 10).

Now the "inverse problem" for singular l^p-complemented copies in function spaces L^F is solved:

THEOREM 3.6. *Let $1 < \alpha \leqq \beta < \infty$ and H be an arbitrary closed subset of the interval $[\alpha, \beta]$. Then there exists an Orlicz function space $L^F(0, 1)$ with indices $\alpha_F^\infty = \alpha$ and $\beta_F^\infty = \beta$, which contains a complemented copy of l^p if and only if $p \in H \cup \{2\}$. Furthermore, for each $p \in H$ the function t^p is not equivalent to any function in $E_{F,1}^\infty$.*

The proof is analogous to the one given in Theorem 2.3 for sequence spaces. We need to consider here the functions F_p defined in the above Theorem 3.5,

$$F_p(t) = t^p \exp\{qf(\log t)\}, \qquad \text{if } t \geqq 1,$$

and to use the method of Theorem 7 in [3] (we omit the details).

REMARK. We do not know whether for every Orlicz function space $L^F(0, 1)$ (resp. sequence space l^F) the set of values $p > 1$ for which the space $L^F(0, 1)$ (resp. l^F) contains a *singular* l^p-complemented copy is always closed.

REFERENCES

1. F. L. Hernández and V. Peirats, *Orlicz function spaces without complemented copies of l^p*, Isr. J. Math. **56** (1986), 355–360.

2. F. L. Hernández and V. Peirats, *Weighted sequence subspaces of Orlicz function spaces isomorphic to l^p*, Arch. Math. **50** (1988), 270–280.

3. F. L. Hernández and B. Rodriguez-Salinas, *On l^p-complemented copies in Orlicz spaces*, Isr. J. Math. **62** (1988), 37–55.

4. W. B. Johnson, B. Maurey, G. Schechtmann and L. Tzafriri, *Symmetric structures in Banach spaces*, Memoirs Am. Math. Soc. **217** (1979).

5. M. Kadec and A. Pelczynski, *Bases, lacunary sequences and complemented subspaces in the spaces L^p*, Studia Math. **21** (1962), 161–176.

6. N. J. Kalton, *Orlicz sequence spaces without local convexity*, Math. Proc. Cambridge Philos. Soc. **81** (1977), 253–277.

7. K. Lindberg, *On subspaces of Orlicz sequence spaces*, Studia Math. **45** (1973), 119–146.

8. J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces II*, Isr. J. Math. **11** (1972), 355–379.

9. J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces III*, Isr. J. Math. **14** (1973), 368–389.

10. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces*, Springer-Verlag, Berlin, 1977.

11. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II. Function Spaces*, Springer-Verlag, Berlin, 1979.

12. W. Luxemburg, *Banach function spaces*, Thesis, Assen, 1955.

13. L. Schwartz, *Analyse: Topologie générale et analyse fonctionnelle*, Hermann, Paris, 1970.